HOSSAM GHANEM

(28) 4.5 Summary of Graphical Methods(A)

Example 1

23 April 27,2000

Let f be a differentiable function on $(-\infty,\infty)$ with

$f(-2)=-1$, $f(-1)=-2$, $f(0)=0$, $f(1)=2$, $f(2)=1$, $\lim_{x\to\pm\infty} f(x)=0$

and

Interval	$(-\infty,-2)$	$(-2,-1)$	$(-1,0)$	$(0,1)$	$(1,2)$	$(2,\infty)$
Sing of $f'(x)$	$-$	$-$	$+$	$+$	$-$	$-$
Sign of $f''(x)$	$-$	$+$	$+$	$-$	$-$	$+$

Answer the following (1 point each)

(a) What are the intervals on which f is increasing, and intervals on which f is decreasing ?

(b) What are the local extrema of f , if any ?

(c) Determine the intervals on which the graph of f is concave upward and the intervals on which the graph is concave downward.

(d) What are the points of inflection of the graph of f , if any ?

(e) Sketch a graph of f indicating local extrema, inflection points, concavity, and asymptotes.

Solution

(a)

$f \searrow \ on\ (-1,1)$

$f \nearrow \ on\ (-\infty,-1)\cup(1,\infty)$

(b)

Maximum local extrema at $x=1$ at $(1,2)$

Minimum local extrema at $x=-1$ at $(-1,-2)$

(c)

Graph f CU on $(-2,0)\cup(2,\infty)$

Graph f CD on $(-\infty,-2)\cup(0,2)$

(d)

Inflection point

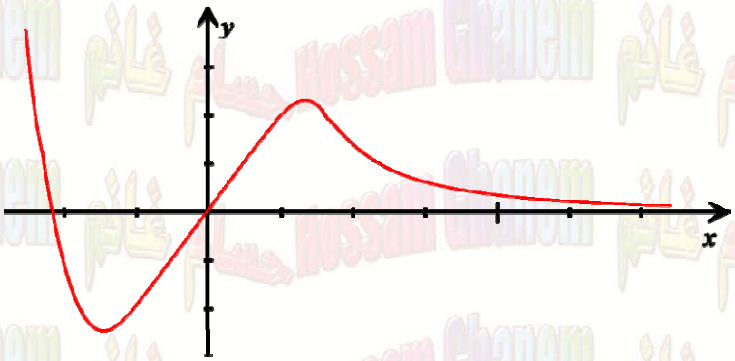
$x=-2 \rightarrow (-2,-1)$

$x=0 \rightarrow (0,0)$

$x=2 \rightarrow (2,1)$

Asymptotes

$y=0$



Example 2

37 May 4, 2006

Let $f(x)=\dfrac{2x^2-2x+1}{x^2}$,and given that $f'(x)=\dfrac{2x-2}{x^3}$ and $f''(x)=\dfrac{6-4x}{x^4}$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

(8 pts.)

Solution

(a)

$H.A:$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{2x^2-2x+1}{x^2}=\lim_{x\to\infty}\frac{2-\frac{2}{x}+\frac{1}{x^2}}{1}=2$$

$$\lim_{x\to-\infty} f(x)=2$$

$\therefore y=2$ $H.A$

$V.A:$

$$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}\frac{2x^2-2x+1}{x^2}=\infty$$

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^-}\frac{2x^2-2x+1}{x^2}=\infty$$

$\therefore x=0$ $V.A$

(b)

$$f'(x)=\frac{2x-2}{x^3}=\frac{2(x-1)}{x^3}$$

$\therefore$ $f\nearrow$ on $(-\infty,0)\cup(1,\infty)$

$f\searrow$ on $(0,1)$

$$f'(x)=0$$

$$x-2=0 \quad \Rightarrow \quad x=1$$

$$f(1)=\frac{2-2+1}{1}=1$$

Minimum local at $(1,1)$

	0		1	
$x-1$	$-$	$-$	$+$	1
x^3	$-$	$+$	$+$	0
$f'(x)$	$\oplus$	$\ominus$	$\oplus$	
	$\nearrow$	$\searrow$	$\nearrow$	

$(c)\quad f''(x)=\dfrac{6-4x}{x^4}=\dfrac{4\left(\frac{3}{2}-x\right)}{x^4}$

The graph of f CU on $\left(-\infty,\frac{3}{2}\right)/\{0\}$

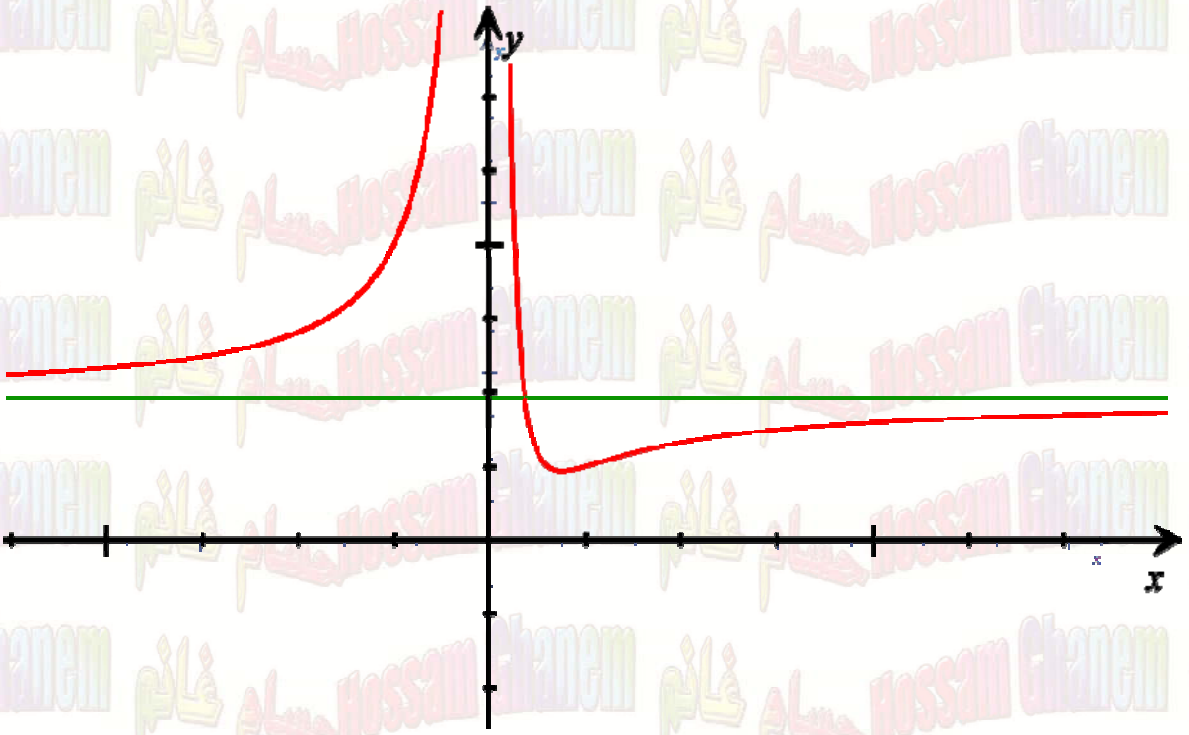
The graph of f CD on $\left(\frac{3}{2},\infty\right)$

$f''(x)=0 \quad \rightarrow 4\left(\frac{3}{2}-x\right)=0 \quad \rightarrow x=\frac{3}{2}$

	0		$\frac{3}{2}$	
$\frac{3}{2}-x$	+	+	−	$\frac{3}{2}$
x^4	+	+	+	0
$f''(x)$	⊕	⊕	⊖	
	CU	CU	CD	

$f\left(\frac{3}{2}\right)=\dfrac{2\left(\frac{9}{4}\right)-2\left(\frac{3}{2}\right)+1}{\frac{9}{4}}=\dfrac{\frac{9}{2}-3+1}{\frac{9}{4}}=\dfrac{18-12+4}{9}=\dfrac{10}{9}$

$\therefore$ at $\left(\frac{3}{2},\frac{10}{9}\right)$ inflection point



Example 3

38 July 17, 2006

Let $f(x)=\dfrac{3-x}{(x-1)^2}$ .and given that $f'(x)=\dfrac{x-5}{(x-1)^3}$ and $f''(x)=\dfrac{2(7-x)}{(x-1)^4}$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

(e) Find the maximum and the minimum values of on [2 , 4].

(10 points)

Solution

(a)

$H.A:$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{3-x}{(x-1)^2}=0$$

$$\lim_{x\to-\infty} f(x)=0$$

$\therefore\ y=0 \quad H.A$

$V.A:$

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}\frac{3-x}{(x-1)^2}=\infty$$

$$\lim_{x\to1^+} f(x)=\lim_{x\to1^+}\frac{3-x}{(x-1)^2}=\infty$$

$\therefore x=1 \quad V.A$

(b)

$\therefore f\ \nearrow\ on\ (-\infty,1)\cup(5,\infty)$

$f\ \searrow\ on\ (1,5)$

$$f'(x)=\frac{x-5}{(x-1)^3}$$

$f'(x)=0 \quad\rightarrow\quad x-5=0 \quad\rightarrow\quad x=5$

$$f(5)=\frac{3-5}{(5-1)^2}=\frac{-2}{16}=\frac{-1}{8}$$

$\therefore$ at $\left(5,\dfrac{-1}{8}\right)$ Minimum local extrema

	1		5	
$x-5$	$-$	$-$	$+$	5
$(x-1)^3$	$-$	$+$	$+$	1
$f'(x)$	$\oplus$	$\ominus$	$\oplus$	
	$\nearrow$	$\searrow$	$\nearrow$	

(c)

The graph of f CU on $(-\infty, 7)/\{1\}$

The graph of f CD on $(7, \infty)$

$$f''(x) = \frac{2(7-x)}{(x-1)^4}$$

$f''(x) = 0 \quad \Rightarrow \quad 7 - x = 0$

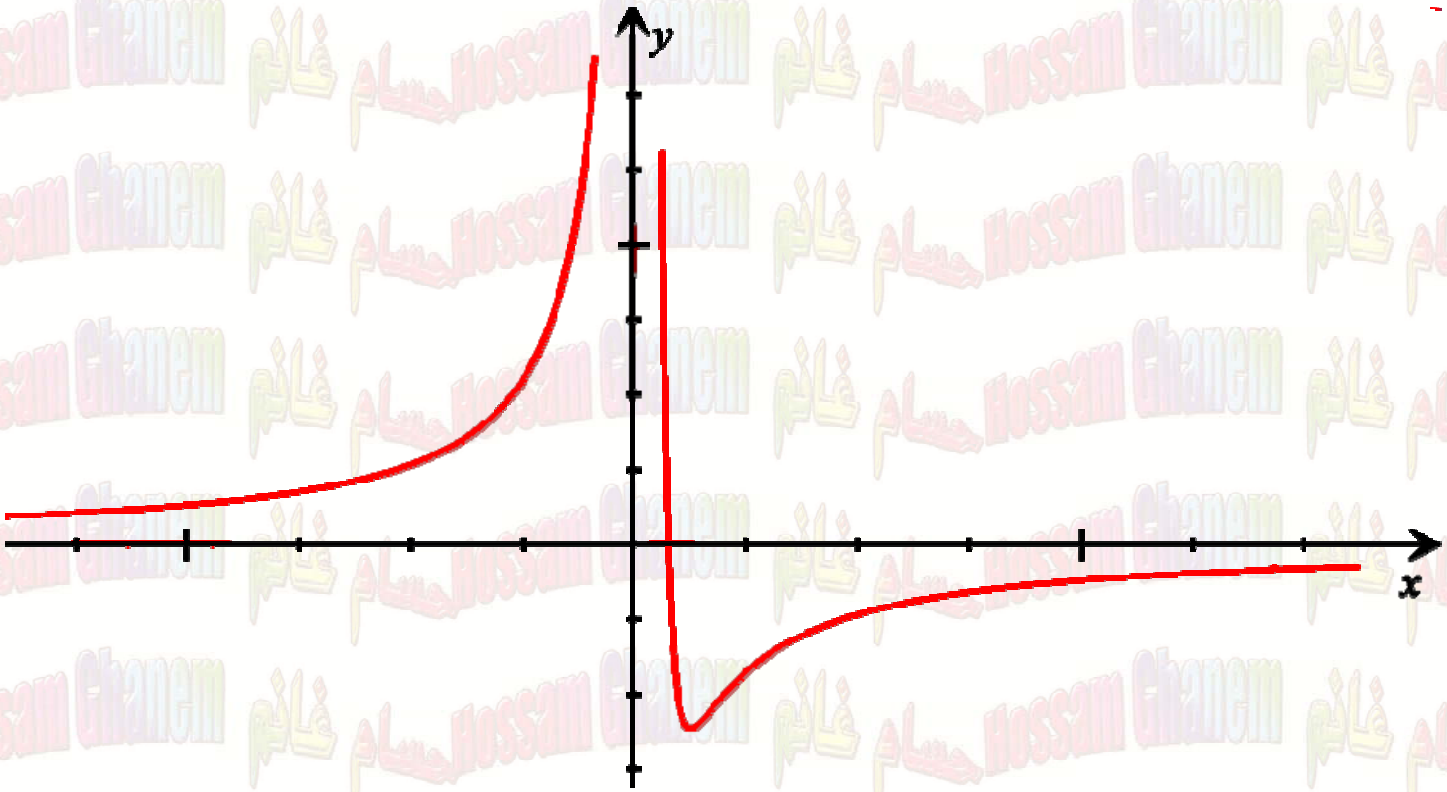
$\therefore x = 7$

$$f(7) = \frac{3-7}{(7-1)^2} = \frac{-4}{36} = \frac{-1}{9}$$

$\therefore$ at $\left(7, \frac{-1}{9}\right)$ inflection point

	1		7	
$7-x$	+	+		−
$(x-1)^4$	+	+		+
$f''(x)$	⊕	⊕		⊖
	CU	CU		CD

(d)



(e)

$$f(2) = \frac{3-2}{(2-1)^2} = 1$$

$$f(4) = \frac{3-4}{(4-1)^2} = \frac{-1}{9}$$

$\therefore$ Maximum V is 1 at $x = 2$

Maximum V is $\frac{-1}{8}$ at $x = 5$

Example 4

39 December 14, 2006

Let $f(x)=\dfrac{x}{x^2-1}$ and given that $f'(x)=-\dfrac{x^2+1}{(x^2-1)^2}$ and $f''(x)=\dfrac{2x\,(x^2+3)}{(x^2-1)^3}$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

(8 pts.)

Solution

(a)

$H.A$:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x}{x^2-1}=0$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x}{x^2-1}=0$$

$$\therefore y=0 \quad H.A$$

$V.A$:

$$\lim_{x\to-1^-} f(x)=\lim_{x\to-1^-}\frac{x}{x^2-1}=\lim_{x\to-1^-}\frac{x}{(x-1)(x+1)}=-\infty$$

$$\lim_{x\to-1^+} f(x)=\lim_{x\to-1^+}\frac{x}{x^2-1}=\lim_{x\to-1^+}\frac{x}{(x-1)(x+1)}=\infty$$

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}\frac{x}{(x-1)(x+1)}=-\infty$$

$$\lim_{x\to1^+} f(x)=\lim_{x\to1^+}\frac{x}{(x-1)(x+1)}=\infty$$

$$\therefore x=-1 \ \&\ x=1 \quad V.A$$

(b)

$f'(x)<0 \quad \forall\, x\in\mathcal{R}/\{-1,1\}$

$\therefore f\searrow$ on $\mathcal{R}/\{-1,1\}$

$f'(x)=-\dfrac{x^2+1}{(x^2-1)^2} \quad \Rightarrow \quad \therefore f'(x)\neq 0$

No local extrema

f has infinite discont at $x=\pm1$

(c) $f''(x)=\dfrac{2x\,(x^2+3)}{(x^2-1)^3}$

$$=\frac{2x\,(x^2+3)}{(x-1)^3(x+1)^3}$$

$\therefore$ graph f CD on $(-\infty,-1)\cup(0,1)$

graph f CU on $(-1,0)\cup(1,\infty)$

$f''(x)=0 \quad \rightarrow \quad 2x\,(x^2+3)=0$

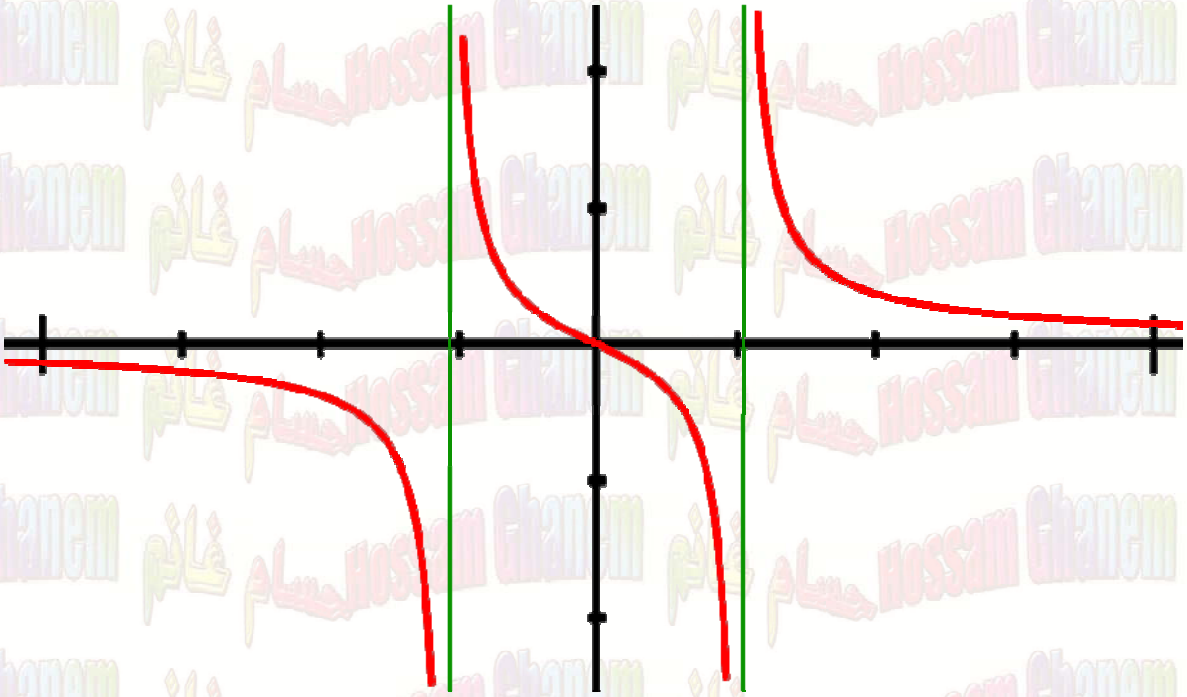
$x=0$

$f(0)=0$

$\therefore$ inflection point at $(0,0)$

	-1		0		1	
x	$-$	$-$		$+$	$+$	0
$(x-1)^3$	$-$	$-$		$-$	$+$	1
$(x+1)^3$	$-$	$+$		$+$	$+$	-1
	$\ominus$	$\oplus$		$\ominus$	$\oplus$	
	CD	CU		CD	CU	

(d)



Homework

1 24 July 20th , 2000

6. Let $f(x) = \dfrac{x^2}{4 - x^2}$

(a) Find the vertical and horizontal asymptotes (if any).

(b) Show that $f'(x) = \dfrac{8x}{(4 - x^2)^2}$

(c) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema (if any).

(d) Given that $f''(x) = \dfrac{32 + 24x^2}{(4 - x^2)^3}$. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward.

(e) Find the point of inflection (if any).

(f) Sketch the graph of f .

2 27 August 2, 2001

7. Let f be a differentiable function such that :

(a) $f(-1) = 0$, $f(0) = 3$, $f(1) = 2$ and $f(3) = 1$

(b) $f'(x) > 0$ on $(-\infty , 0) \cup (3 , \infty)$ and $f'(x) < 0$ on $(0 , 3)$.

(c) $f''(x) < 0$ on $(-\infty , 1)$ and $f''(x) > 0$ on $(1 , \infty)$

Sketch the graph of $y = f(x)$, showing the local extrema and the points of inflection if any.

Homework

3

36 January 17, 2010

5. (4 pts.) Sketch a graph of $f(x)$ that satisfies ALL of the following conditions

$f(0) = 0$	
$\lim_{x\to-\infty} f(x) = 2$	$\lim_{x\to+\infty} f(x) = +\infty$
$\lim_{x\to1^-} f(x) = +\infty$	$\lim_{x\to1^+} f(x) = -\infty$
$f'(x) < 0$ on $(-\infty, 0)$	$f'(x) > 0$on $(0, 1)$and $(1, +\infty)$
$f''(x) < 0$ on $(-\infty, 0)$and $(1, +\infty)$	$f''(x) > 0$ on $(0, 1)$

4

15 February 12.1996

3. Let $f(x) = \dfrac{x^4 + 16}{x^2}$

(a) Find the intervals on which f is increasing or is decreasing , and find the local extrema of f (if any) .

(b) Find the intervals on which the graph of f is concave upward or concave downward. and find the points of inflection (if any) .

(c) Find the vertical and horizontal asymptotes for the graph of f (if any) .

(d) Sketch the graph of f. (2+2+2+2 points)

5

18 May 24 ,2000

Let $f(x) = x^3 + 3x^2 - 9x + 1$.

(a) (3 pts.) Find the local extreme of f.

(b) (3 pts.) Find the intervals on which the graph of f is concave upward. What are the points of inflection?